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Redefining Approaches for Engaging English Learners With Mathematical Ideas

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The study of secondary mathematics involves many ideas, but too often high school mathematics curriculum materials and instruction begin by giving students the vocabulary first, along with definitions, before students have been able to explore the ideas. Particularly for students bureaucratically designated as English Learners, preteaching definitions can actually hinder their understanding rather than enabling them to quickly and deeply explore important mathematical ideas (e.g., de Araujo & Smith, 2022).

In the interest of developing a more productive approach, this brief considers how definitions are currently introduced and used in mathematics education at the secondary level. The brief addresses the following questions with the aim of helping educators ensure that English Learners can participate fully in mathematical learning:

- What challenges do teachers need to consider for using good mathematical definitions with English Learners?
- How do curriculum materials currently approach the definition of key terms?
- How can learning opportunities focus on ideas that enable English Learners to explore and generate their own definitions?

The Dilemma of Definitions for Key Terms and Concepts

As secondary English Learners explore important mathematical ideas, their teachers will need to “use correct and useful definitions” in designing and enacting instruction (Ball et al., 2008). For English Learners, it is particularly critical for teachers to balance two needs that may compete for time and attention: (a) to enable English Learners to access and explore ideas deeply as soon possible and (b) to guide English Learners to develop, use, or apply definitions that are more formal and are suitable for engaging in deductive reasoning.

This brief’s central claim is that to be useful, an initial definition of a mathematical term or concept needs to be “correct” enough to launch English Learners’ explorations of ideas. Definitions that are more formal ideally emerge after significant opportunities to engage with underlying ideas and as the basis for further deductive reasoning that is at the heart of mathematics.

Of course, the dilemma of definitions in mathematics extends beyond English Learners. In fact, when it comes to mathematics, it could be argued that the “stipulative” (e.g., Edwards & Ward, 2008) nature of some definitions of formal mathematical concepts is unique, in contrast to the “extracted” nature of how words are actually used in practice. As a discipline, mathematics differs from endeavors that are more empirical in its formal study of objects whose properties are axiomatically stipulated. For English Learners, it will be critical to make explicit links—and contrasts—between the more everyday and elaborated uses of terms and their formal definitions, which may simultaneously be highly technical and minimalist by design.

Definitions in Current Curricula

To better understand the present nature of definitions in commonly used curricula at the 8th and 9th grade levels, the authors of this brief conducted a scan of the three textbooks most commonly adopted by districts in California (Chu et al., 2022). These three textbooks are *College Preparatory Mathematics* (*CPM*) (College Preparatory Math [CPM] Educational Program, 2015, 2016), *Big Ideas* (Larson & Boswell, 2019), and *Go Math!* (Houghton Mifflin Harcourt, 2014). To get a sense of the potential variation across the three subdomains of secondary mathematics, key topics were selected from across algebra (linear functions), geometry (congruence), and data/statistics (bivariate correlation). The scan drew upon recent work in mathematics education that contrasts between curriculum as a “delivery mechanism” and as a “thinking device” (Choppin et al., 2022). When curriculum functions as a “delivery mechanism,” key definitions are given at the beginning rather than developed along the way. Indeed, three trends were salient in the scan of textbooks:

- Definitions are often given before ideas are explored and are defined in vague ways that are never formalized.
- Definitions are often overly verbose, offering information that could be deduced from a more minimal definition.
- Definitions in 9th grade are very similar to those in 8th grade rather than demonstrating greater sophistication.

Vague Definitions Not Formalized

Within the domain of statistics, terms such as “cluster” and “outlier” are defined by textbooks in advance of any examples. Further, the actual definitions given are neither mathematically precise nor particularly accessible. In *Go Math!*, for example, a *cluster* is defined as “a set of closely grouped data,” and an *outlier* is defined as “a data point that is very different from the rest of the data in the set” (Houghton Mifflin Harcourt, 2014, p. 434). Although these definitions may be broad enough to invite students to initially investigate the ideas, they do not offer students clear criteria for deciding whether data are clustered or outliers, and more formal criteria are never developed in the textbooks reviewed for this brief.

Verbose Definitions Telling Properties

Unlike the statistics definitions that are imprecise or vague in the textbooks that were reviewed, many of the definitions given of slope and congruence are overly technical and, simultaneously, verbose. For example, the *CPM* definition of “slope” is as follows:

> *A ratio that describes how steep (or flat) a line is. Slope can be positive, negative, or even zero, but a straight line has only one slope. .... When the equation of a line is written in* $y = mx + b$ *form,* m *is the slope of the line. A line has a positive slope if it slopes upward from left to right on a graph, a negative slope if it slopes downward from left to right, zero slope if it is horizontal, and an undefined slope if it is vertical. Slope is interpreted in context as the amount of change in the* y*-variable for an increase of one unit in the* x *variable. Some texts refer to slope as the ratio of the “rise over the run.”*

Only at the end of this long paragraph is the essence of slope—as a unit rate of change—mentioned (Stump, 1999; Hamburger & Chu, 2019). The barrage of information contained within the definition is unlikely to enhance English Learners’ access to the concept of slope and, more importantly, their understanding of what is essential (the idea of unit rate of change) as opposed to what is more incidental or tangential (the specific contexts, representations, and qualitative interpretations).

Repetitive Definitions and Concepts

Comparing the 8th grade curriculum to 9th grade integrated mathematics courses in *CPM*, there is a great deal of repetition—both in topics and in how terms are defined. For example, the definition of congruent shapes is nearly identical across the grades (see Table 1).

Table 1. Definition of Congruent Shapes

8th grade	9th grade
"Two shapes are congruent if there is a sequence of rigid transformations that carries one onto the other. The corresponding angles and sides of congruent polygons have equal measures. Congruent shapes are similar and have a scale factor of 1."	**"Two shapes are congruent if there is a sequence of rigid transformations that carries one onto the other. The corres-ponding angles and sides of congruent polygons have equal measures. Congruent shapes are similar and have a scale factor of 1. The symbol for congruent is $\cong$."**

The main exception to this repetition of content is the introduction of the notion of "least squares regression line" in statistics in the 9th grade as a more technical approach to the idea of "line of best fit" introduced in the 8th grade curriculum materials. This approach, however, is lacking in another way—given the heavy reliance on calculators to generate such lines, it is not clear that students have adequate opportunity to understand *why* such lines fit best.

A More Productive Approach to Definitions

Given this current state of affairs, what is a more productive way forward? The textbooks that were reviewed present ideas and terms as accomplished facts rather than as notions that students can explore and, through their explorations, begin to define. For students to be more central participants in the development of conceptual understanding in mathematics, their experiences need to be fundamentally transformed. Teachers need to focus on ideas and examples first and have students then engage in *defining as a process* to arrive at definitions that are more elegant, formal, and meaningful over time.

Broadly speaking, teachers can consider how learners negotiate and refine the "concept images" they use to make meaning and connections (Vinner, 2002). Such concept images offer the working examples and illustrative instances (e.g., mental images of squares) that can serve as the basis for more formality in verbal descriptions of essential properties. It may be more productive to help students explore different "example spaces" from which they can later articulate a concise and elegant description of key properties (e.g., Watson & Mason, 2005). As students engage in an ongoing process of defining, they can develop or explore what has been termed "simple but deep" definitions (Molina, 2012) or working definitions (Hamburger & Chu, 2019) that are just enough to offer access to ideas for deeper exploration. For example, students may explore the concept of arithmetic mean or average as a point of balance and then, later on, connect this idea to the standard algorithm for calculating the mean or average, a procedure which is often conflated with the definition.

By starting with concept images and example spaces, it is natural for the role of definitions to change over time in a classroom. To provide structure for this process of rethinking how definitions of key ideas are handled in secondary mathematics, a bounded instructional sequence known as Three Moment Architecture may be helpful (Walqui, 2019; Hamburger & Chu, 2019). Specifically, during a lesson that may stretch over several class periods, there are three distinct "moments" that reflect different pedagogical purposes and learning opportunities:

1. Preparing the Learners
2. Interacting With the Concept
3. Extending Understanding

The purpose of Preparing the Learners is to connect students' prior knowledge (not necessarily prerequisite knowledge) with the main ideas of the lesson. During this moment, the teacher may introduce a few key terms for the lesson, with supporting context and examples. Then, in Interacting With the Concept, students explore ideas and concepts—using appropriate terms along the way—across multiple representations, procedures, and situational contexts or metaphors. Once they have consolidated their understanding, students in the Extending Understanding moment apply those ideas to new scenarios, connect with other ideas, or re-present their understandings in new genres. The following sections elaborate upon the process of defining in each of these moments and ground the examples in the topic of statistics, focusing on the idea of outliers within the context of bivariate data.

Defining Ideas While Preparing the Learners

For the mathematical topic of *outliers*, it would be productive to offer students a broad array of data and distributions, some with and others without outliers. The array of data can also include borderline cases of outliers in a variety of representations, including numerical and graphical ones. The data sets chosen can also include the basis for other statistical concepts, such as clusters or correlated sets of data. If asked what they notice and wonder, students can generate observations and questions that are more authentic than if they were just presented with a predetermined definition. Indeed, students may also generate labels or ideas that extend beyond the target term "outlier." Students tend to be able to generate descriptions and criteria for outliers that are at least as helpful as those provided in textbooks.

Applying Definitions and Discovering Properties While Interacting With Concepts

That initial working definition enables access to the idea that students can then explore in connection with other ideas in the Interacting With the Concept moment of a lesson. For bivariate data represented as scatterplots, this exploration may include making connections with univariate measures of center and distribution that they have studied before. For example, students might explore how the means of the two variables considered independently are related to ideas of center of gravity and the centroid. Students could also explore how the medians, calculated for each variable separately, are related to the overall distribution of data on a scatterplot. As they explore how to represent these univariate measures jointly on the scatterplot, students can begin to articulate criteria for identifying outliers. Students can then begin to put to the test some of the working definitions they have generated, as students can consider how the outlier relates to measures of center (mean and median) as well as spread (standard deviation and interquartile range). Given the diversity of approaches, such explorations may lend themselves to a jigsaw student-grouping structure in which different groups become experts on how outliers relate to different statistical measures (U.S. Department of Education, Office of English Language Acquisition, 2020).

Formalizing Definitions to Extend Understanding

After English Learners have had sufficient opportunities to interact with their peers in exploring these ideas, they can begin to formalize their definitions as they try to extend their working understandings to apply to a more general range of cases. This formalizing could take on two forms: Students could develop their own more formal definitions, or they could critically examine other more technical definitions that they are offered. For outliers, there are different ways that students can apply either interquartile ranges or standard deviation or error to generate definitions that are more quantitatively specific. Students can also apply commonly accepted definitions, such as the univariate criterion that any data point more than 1.5 times the interquartile range from the third or first quartile is an outlier. These questions offer students opportunities to revisit and refine their initial thinking, applying the statistical tools that they have developed.

What Can Mathematics Educators Do?

Beyond using or building on this particular example, the following are some concrete, specific actions that mathematics educators can take as they consider the role of defining and of definitions in the learning opportunities they plan for English Learners:

- Think carefully about the mathematical ideas, not the words used to refer to the ideas. Think about what working definition would maximize student access to those ideas without giving them too much detail or telling them what they could discover on their own.
- Early in the lesson, engage students in defining, in which they come up with working criteria about key ideas in an emerging process of generating working definitions and then continually refining those definitions.
- Create activities that enable students to explore ideas by connecting multiple representations, all with the purpose of uncovering relationships and connections between the new concept and definition and other mathematical ideas.
- Invite and support students to extend their understanding to definitions that are more formal; doing so offers English Learners an opportunity to revisit and revise their initial thinking in light of their learning as well as to engage in forms of reasoning that are more deductive.
- Nurture students' understanding of the genre of definition and the role it plays in mathematics as a deductive discipline.

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